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MEAN DIMENSION OF THE DYNAMICAL SYSTEM OF BRODY CURVES

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Abstract. Mean dimension measures the size of an infinite dimensional dynamical system. Brody curves are one-Lipschitz entire holomorphic curves in the projective space, and they form a topological dynamical system. Gromov started the problem of estimating its mean dimension in the paper of 1999. We solve this problem. Namely we prove the exact mean dimension formula of the dynamical system of Brody curves. Our formula expresses the mean dimension by the energy density of Brody curves. The proof is based on a novel application of the metric mean dimension theory of Lindenstrauss and Weiss.

1. Mean dimension formula

Let $z = x + \sqrt{-1}y$ be the standard coordinate in $\mathbb{C}$. Let $f = [f_0 : f_1 : \cdots : f_N] : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic curve (f_i: holomorphic function). We define the **spherical derivative** $|df|(z) \geq 0$ by

$$|df|^2(z) := \frac{1}{4\pi}\Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2) \quad \left(\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This is the local Lipschitz constant. For a unit tangent vector $u \in T_z\mathbb{C}$ the Fubini–Study length of $df(u) \in T_{f(z)}\mathbb{C}P^N$ is equal to the spherical derivative $|df|(z)$. A holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ is called a Brody curve if it is 1-Lipschitz, namely $|df| \leq 1$ all over the plane. The name comes from the work of Brody [2]. He proved that a projective variety is Kobayashi hyperbolic if and only if it does not contain a non-constant Brody curve. (See Duval [5] for a much deeper version of this *Bloch–Brody principle.*) The original motivation for the study of Brody curves comes from this connection to the hyperbolicity.

Recently new waves have begun, and Brody curves attract new interests of several authors [1, 3, 4, 5, 6, 12, 14, 15, 19]. One source of new interests is the work of Gromov [8]. He introduced a new topological invariant of dynamical systems called **mean dimension**, and proposed a program to study many infinite dimensional dynamical systems in geometric analysis from the viewpoint of this invariant. *The dynamical system consisting of Brody curves* is the simplest example in this program. The purpose of our paper is

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to show that this perspective reveals a new fundamental structure in holomorphic curve theory.

We define $\mathcal{M}(\mathbb{C}P^N)$ as the space of all Brody curves $f : \mathbb{C} \to \mathbb{C}P^N$ endowed with the compact-open topology. This is compact and the group $\mathbb{C}$ continuously acts on it by

$$\mathbb{C} \times \mathcal{M}(\mathbb{C}P^N) \to \mathcal{M}(\mathbb{C}P^N), \quad (a, f(z)) \mapsto f(z+a).$$

Our main object is the dynamics of this action. The space $\mathcal{M}(\mathbb{C}P^N)$ is infinite dimensional, and the topological entropy of the $\mathbb{C}$-action is also infinite. So $\mathcal{M}(\mathbb{C}P^N)$ is very large. Mean dimension is precisely an invariant introduced for this kind of large dynamical systems. Its definition is reviewed in Subsection 2.1. Here we explain intuition about this concept. A basic example is the shift action of $\mathbb{Z}$ on the Hilbert cube $[0,1]^{\mathbb{Z}}$. This is infinite dimensional and of infinite topological entropy. Its mean dimension is one. Roughly speaking, this means that the system $[0,1]^{\mathbb{Z}}$ has one parameter per every unit second. In general mean dimension counts the number of parameters "averaged by a group action".

The significance of mean dimension in topological dynamics was clarified by the works of Lindenstrauss and Weiss [11, 10]. Here is a sample of their results [10, Theorem 5.1]. Let X be a compact metric space with a continuous $\mathbb{Z}$-action. If X can be equivariantly embedded into the system $[0,1]^{\mathbb{Z}}$ then its mean dimension is less than or equal to one. Conversely, if X is minimal (i.e. every orbit is dense) and its mean dimension is less than 1/36, then it can be equivariantly embedded into $[0,1]^{\mathbb{Z}}$. This striking embedding theorem shows that mean dimension properly measures the size of an infinite dimensional dynamical system.

We denote by $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$ the mean dimension of the $\mathbb{C}$-action on the space of Brody curves $\mathcal{M}(\mathbb{C}P^N)$. This is a nonnegative real number which counts the number of parameters in $\mathcal{M}(\mathbb{C}P^N)$ "per every unit area of the plane $\mathbb{C}$".

Problem 1.1 (Main problem). Compute the mean dimension $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$.

This problem was started by Gromov himself [8, p. 396 (c)]. He asked what kind of properties are reflected on the estimate of the mean dimension. In our notation, his result in [8, p. 396 (c)] is the upper bound

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq 4N. \tag{1.1}$$

The proof is based on the Nevanlinna theory (Eremenko [6, Theorem 2.5]). In a series of works [14, 16, 12], we have been sharpening Gromov's estimate. In the present paper we finally reach the answer to the problem.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. We define its **energy density** $\rho(f)$ by

$$\rho(f) := \lim_{R\to\infty} \left(\frac{1}{\pi R^2} \sup_{a\in\mathbb{C}} \int_{|z-a|<R} |df|^2 dxdy \right). \tag{1.2}$$

This limit always exists (see Section 2.2). It evaluates how densely the energy is distributed over the plane. We define $\rho(\mathbb{C}P^N)$ as the supremum of $\rho(f)$ over $f \in \mathcal{M}(\mathbb{C}P^N)$. It is known that ([15])

$$0 < \rho(\mathbb{C}P^N) < 1, \quad \lim_{N\to\infty} \rho(\mathbb{C}P^N) = 1.$$

The main theorem is the formula expressing the mean dimension by the energy density.

Theorem 1.2 (**Main theorem**).

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) = 2(N+1)\rho(\mathbb{C}P^N).$$

We briefly review preceding researches. Gromov's upper bound (1.1) is the first result on $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$. In [14] we proved an improved upper bound

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \le 4N\rho(\mathbb{C}P^N). \tag{1.3}$$

The proof again used the Nevanlinna theory. The lower bound on the mean dimension was developed in [16, 12]. The main theorem of [12] is the sharp lower bound

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \ge 2(N+1)\rho(\mathbb{C}P^N). \tag{1.4}$$

Luckily this lower bound coincides with the upper bound (1.3) when $N = 1$. So we got the formula ([12, Corollary 1.2])

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4\rho(\mathbb{C}P^1).$$

This was the first nontrivial calculation of the mean dimension in geometric analysis. When $N \ge 2$, the upper bound (1.3) and lower bound (1.4) still have a gap. The purpose of the present paper is to fulfill this gap by proving the sharp upper bound

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \le 2(N+1)\rho(\mathbb{C}P^N). \tag{1.5}$$

Combined with the lower bound (1.4), this establishes Theorem 1.2.

The proof of the sharp upper bound (1.5) requires an idea completely different from Gromov's upper bound (1.1) and our previous bound (1.3). Before explaining a new idea, we review a previous argument. Most known upper bounds on mean dimension are based on the techniques of **sampling and embedding** (Gromov [8, Chapters 3 and 4]). The proof of (1.3) is a typical one, and it goes as follows. Take a lattice $\Gamma \subset \mathbb{C}$ satisfying $\mathrm{Area}(\mathbb{C}/\Gamma) < 1/(2\rho(\mathbb{C}P^N))$. By using Nevanlinna's first main theorem, we can prove that the *sampling map*

$$\mathcal{M}(\mathbb{C}P^N) \to (\mathbb{C}P^N)^\Gamma, \quad f \mapsto f|_\Gamma = (f(\gamma))_{\gamma\in\Gamma},$$

is an embedding. Then we get the upper bound on the mean dimension with respect to the Γ-action

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \Gamma) \le \dim((\mathbb{C}P^N)^\Gamma : \Gamma) = 2N.$$

The mean dimension of the $\mathbb{C}$-action follows from this by

$$\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) = \dim(\mathcal{M}(\mathbb{C}P^N) : \Gamma)/\mathrm{Area}(\mathbb{C}/\Gamma) \leq 2N/\mathrm{Area}(\mathbb{C}/\Gamma).$$

Letting $\mathrm{Area}(\mathbb{C}/\Gamma) \to 1/(2\rho(\mathbb{C}P^N))$, we get (1.3). This proof is very simple but, unfortunately, non-flexible. It is hopeless to prove the optimal bound (1.5) by this method, and there also exist several problems facing similar difficulties. For example, Gromov [8, Chapter 4] studied a dynamical system consisting of complex subvarieties in $\mathbb{C}^N$, and he proved an upper bound on the mean dimension by using the sampling argument. But his estimate is very far from the conjectural value [8, p. 408 Corollary, p. 409 Remark, p. 409 Remarks and open questions (a)]. Sampling method seems to be inadequate for obtaining a precise estimate.

In order to overcome these difficult situations, we started to develop a completely new approach in [18]. The paper [18] examined a new technique in the context of Yang–Mills gauge theory. Based on this experience, now we attack to Brody curves. A key novel ingredient of our approach is **metric mean dimension** introduced by Lindenstrauss–Weiss [11]. This is a geometric/information theoretic version of mean dimension. Its idea is as follows. Given a dynamical system, suppose we try to store on computer the orbits of the system over a very long period of time to an accuracy of $\varepsilon > 0$. How many memory (bits) do we need? Asymptotically (as the period of time goes to infinity and ε goes to zero) the answer is given by metric mean dimension. The precise definition is given in Subsection 2.1.

A fundamental theorem of Lindenstrauss–Weiss asserts that metric mean dimension is always an upper bound on mean dimension. So we can approach to mean dimension via metric mean dimension. This provides us great flexibility because *metric mean dimension is a more local quantity than mean dimension.* Intuitively speaking, when we store the information of a dynamical system on computer, we can decompose the system into small pieces and try to memorize each piece separately. Namely we can decompose a global problem into local ones which are more suitable for detailed analysis. This enables us to apply the analytic machinery developed in [16, 12] to the problem of proving the upper bound (1.5). The techniques in [16, 12] were originally introduced for the lower bound (1.4). Its (unexpected) usefulness for the upper bound seems to suggest that our method is a right way to the problem. It is very likely that our approach can be also applied to other problems, and we hope that it will grow to be a standard technique in mean dimension theory.

2. Preliminaries

2.1. **Mean dimension.** We review basic definitions of mean dimension here. The main references are Gromov [8] and Lindenstrauss–Weiss [11]. Readers can find further information in Lindenstrauss [10] and Gutman [9].

First we need to introduce some metric invariants. Let (X, d) be a compact metric space. Let Y be a topological space, and $f : X \to Y$ a continuous map. For a positive number ε, we call f an **ε-embedding** if $\mathrm{Diam}(f^{-1}(y)) < \varepsilon$ for all $y \in Y$. This means that f looks like an embedding if we ignore an error smaller than ε. We define the **ε-width dimension** $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there exist an n-dimensional finite simplicial complex P and an ε-embedding from (X, d) to P. The topological dimension is given by

$$\dim X = \lim_{\varepsilon \to 0} \mathrm{Widim}_\varepsilon(X, d).$$

For $\varepsilon > 0$ we set

$$\#(X, d, \varepsilon) := \min\{\, |\alpha| \,|\, \alpha \text{ is an open covering of } X \text{ with } \mathrm{Diam} U < \varepsilon \text{ for all } U \in \alpha\},$$
$$\#_{\mathrm{sep}}(X, d, \varepsilon) := \max\{n \geq 1 |\, \exists x_1, \dots, x_n \in X \text{ with } \mathrm{dist}(x_i, x_j) > \varepsilon \ (i \neq j)\}.$$

Here "sep" is the abbreviation for **separated set**. These two quantities are essentially equivalent to each other: For $0 < \delta < \varepsilon/2$

$$\#_{\mathrm{sep}}(X, d, \varepsilon) \leq \#(X, d, \varepsilon) \leq \#_{\mathrm{sep}}(X, d, \delta). \tag{2.1}$$

The next lemma will be used later. Its proof is trivial

Lemma 2.1. *Let (X, d) and (Y, d') be compact metric spaces. Let $\varepsilon, \delta > 0$. Suppose there exists a map (not necessarily continuous) $f : X \to Y$ satisfying*

$$d(x, y) > \varepsilon \Rightarrow d'(f(x), f(y)) > \delta.$$

Then $\#_{\mathrm{sep}}(X, d, \varepsilon) \leq \#_{\mathrm{sep}}(Y, d', \delta)$.

We will also need the following.

Example 2.2. Let $(V, \|\cdot\|)$ be a real n-dimensional Banach space. Let $B_r(V)$ be the closed r-ball of V around the origin. For any $\varepsilon > 0$

$$\#_{\mathrm{sep}}(B_r(V), \|\cdot\|, \varepsilon) \leq \left(\frac{\varepsilon + 2r}{\varepsilon}\right)^n.$$

Proof. Let μ be the Lebesgue measure (i.e. the Haar measure with respect to the translation) on V normalized by $\mu(B_1(V)) = 1$. For any $r > 0$ we have $\mu(B_r(V)) = r^n$. Choose $\{x_1, \dots, x_N\} \subset B_r(V)$ with $\|x_i - x_j\| > \varepsilon$ for $i \neq j$. Let B_i be the closed $\varepsilon/2$-ball around x_i. These B_i are disjoint and contained in $B_{r+\varepsilon/2}(V)$. Hence

$$N(\varepsilon/2)^n = \mu\left(\bigcup_{i=1}^N B_i\right) \leq \mu(B_{r+\varepsilon/2}(V)) = (r + \varepsilon/2)^n.$$

□

Consider the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $r \geq 0$ we define $D_r(a)$ as the closed r-ball around a in the plane. We abbreviate $D_r(0)$ as D_r. For $\Omega \subset \mathbb{C}$ and $r > 0$ we define $\partial_r \Omega$ as the set of $a \in \mathbb{C}$ such that $D_r(a)$ non-trivially intersects with both Ω and $\mathbb{C} \setminus \Omega$. A sequence $\{\Omega_n\}_{n\geq 1}$ of bounded Borel subsets of $\mathbb{C}$ is called a **Følner sequence** if for any $r > 0$

$$\lim_{n\to\infty} \frac{\mathrm{Area}(\partial_r \Omega_n)}{\mathrm{Area}(\Omega_n)} = 0 \quad (\mathrm{Area}(\cdot) \text{ is the standard Lebesgue measure}).$$

For example $\Omega_n := D_n$ is a Følner sequence. $\Omega_n := [0, n]^2$ is also. The next lemma (which holds in a more general context of amenable groups) is a basis of the definition of mean dimension. This was originally found by Ornstein–Weiss [13, Chapter I, Sections 2 and 3]. This formulation is due to Gromov [8, p. 336].

Lemma 2.3 (Ornstein–Weiss lemma)**.** *Let h : {bounded Borel subsets of $\mathbb{C}$} $\to \mathbb{R}$ be a non-negative function satisfying the following three conditions.*
(1) If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \leq h(\Omega_2)$.
(2) $h(\Omega_1 \cup \Omega_2) \leq h(\Omega_1) + h(\Omega_2)$.
(3) For any $a \in \mathbb{C}$ and any bounded Borel subset $\Omega \subset \mathbb{C}$, we have $h(a + \Omega) = h(\Omega)$ where $a + \Omega := \{a + z \in \mathbb{C} | z \in \Omega\}$.

Then for any Følner sequence Ω_n $(n \geq 1)$ in $\mathbb{C}$, the limit of the sequence

$$\frac{h(\Omega_n)}{\mathrm{Area}(\Omega_n)} \quad (n \geq 1)$$

exists, and its value is independent of the choice of a Følner sequence.

Choose a Følner sequence $\{\Omega_n\}_{n\geq 1}$ in $\mathbb{C}$. Let (X, d) be a compact metric space, and suppose the group $\mathbb{C}$ continuously acts on X. For a subset Ω of $\mathbb{C}$ we define a new distance d_Ω on X by

$$d_\Omega(x, y) = \sup_{a\in\Omega} d(a.x, a.y) \quad (x, y \in X).$$

For any $\varepsilon > 0$ the function $h(\Omega) := \mathrm{Widim}_\varepsilon(X, d_\Omega)$ satisfies the three conditions in the Ornstein–Weiss lemma (Lemma 2.3). Then we define the **mean dimension** $\dim(X : \mathbb{C})$ by

$$\dim(X : \mathbb{C}) := \lim_{\varepsilon\to 0} \left(\lim_{n\to\infty} \frac{\mathrm{Widim}_\varepsilon(X, d_{\Omega_n})}{\mathrm{Area}(\Omega_n)} \right).$$

The value of $\dim(X : \mathbb{C})$ is independent of the choice of a distance d on X compatible with the topology. So the mean dimension is a topological invariant.

Next we introduce **metric mean dimension** (Lindenstrauss–Weiss [11, Section 4]). For any $\varepsilon > 0$ the function $h(\Omega) := \log \#(X, d_\Omega, \varepsilon)$ also satisfies the conditions of the Ornstein–Weiss lemma. Then we define the "entropy at the scale ε" by

$$S(X, d, \varepsilon) := \lim_{n\to\infty} \frac{\log \#(X, d_{\Omega_n}, \varepsilon)}{\mathrm{Area}(\Omega_n)}.$$

The topological entropy $h_{\mathrm{top}}(X:\mathbb{C})$ is the limit of $S(X,d,\varepsilon)$ as $\varepsilon\to 0$. We define the metric mean dimension by

$$\dim_{\mathrm{M}}(X,d:\mathbb{C}) := \liminf_{\varepsilon\to 0}\frac{S(X,d,\varepsilon)}{|\log\varepsilon|}.$$

This *depends* on the choice of a distance. The next theorem is fundamental ([11, Theorem 4.2]).

Theorem 2.4 (Lindenstrauss–Weiss theorem)**.** *Metric mean dimension is an upper bound on mean dimension:*

$$\dim(X:\mathbb{C}) \le \dim_{\mathrm{M}}(X,d:\mathbb{C}).$$

Therefore we can approach to an upper bound on mean dimension via metric mean dimension. A difficulty of mean dimension lies in its *global* nature. We need to construct an ε-embedding from X to a simplicial complex for an upper bound on $\dim(X:\mathbb{C})$. But in many situations (in particular the case of $X=\mathcal{M}(\mathbb{C}P^N)$) the space X is a mysterious infinite dimensional space, and it is highly nontrivial to construct an ε-embedding from X to an appropriate simplicial complex. We can relax this difficulty by using metric mean dimension because metric mean dimension is a more *local* quantity. Its local nature is probably not obvious in the above definition. The next lemma provides an explicit formulation. Here we use the notation $D_R(\Lambda) := \bigcup_{a\in\Lambda} D_R(a)$ for $R\ge 0$ and $\Lambda\subset\mathbb{C}$.

Lemma 2.5. *For any $\delta>0$ and $R\ge 0$ the metric mean dimension* $\dim_{\mathrm{M}}(X,d:\mathbb{C})$ *is equal to*

$$\liminf_{\varepsilon\to 0}\left\{\left(\limsup_{L\to\infty}\frac{\sup_{x\in X}\log\#(B_\delta(x,d_{D_R([0,L]^2)}),d_{[0,L]^2},\varepsilon)}{L^2}\right)/|\log\varepsilon|\right\}. \tag{2.2}$$

Here $B_\delta(x,d_{D_R([0,L]^2)})$ is the closed δ-ball around x with respect to the distance $d_{D_R([0,L]^2)}$. Indeed we can replace $\limsup_{L\to\infty}$ *with* $\lim_{L\to\infty}$*. But we don't need this fact.*

Proof. We denote the right-hand side by $\dim'_{\mathrm{M}}(X,d:\mathbb{C})$. It is obvious that $\dim_{\mathrm{M}}(X,d:\mathbb{C})\ge\dim'_{\mathrm{M}}(X,d:\mathbb{C})$. We can choose $x_1,\dots,x_n\in X$ such that

$$X=\bigcup_{i=1}^n B_\delta(x_i,d_{D_R([0,L]^2)}),\quad n\le\#(X,d_{D_R([0,L]^2)},\delta).$$

Then $\#(X,d_{[0,L]^2},\varepsilon)$ is bounded by

$$\sum_{i=1}^n\#(B_\delta(x_i,d_{D_R([0,L]^2)}),d_{[0,L]^2},\varepsilon)\le n\sup_{x\in X}\#(B_\delta(x,d_{D_R([0,L]^2)}),d_{[0,L]^2},\varepsilon).$$

Therefore

$$\frac{\log\#(X,d_{[0,L]^2},\varepsilon)}{L^2}\le\frac{\log\#(X,d_{D_R([0,L]^2)},\delta)}{L^2}+\frac{\sup_{x\in X}\log\#(B_\delta(x,d_{D_R([0,L]^2)}),d_{[0,L]^2},\varepsilon)}{L^2}.$$

Letting $L \to \infty$, we get

$$S(X,d,\varepsilon) \le S(X,d,\delta) + \limsup_{L\to\infty} \frac{\sup_{x\in X} \log \#(B_\delta(x, d_{D_R([0,L]^2)}), d_{[0,L]^2}, \varepsilon)}{L^2}.$$

Divide this by $|\log \varepsilon|$, and let $\varepsilon \to 0$. Then we get $\dim_{\mathrm{M}}(X, d : \mathbb{C}) \le \dim'_{\mathrm{M}}(X, d : \mathbb{C})$. $\square$

The formula (2.2) probably looks complicated. The point is that we only need to estimate $\#(B_\delta(x, d_{D_R([0,L]^2)}), d_{[0,L]^2}, \varepsilon)$ for the calculation of the metric mean dimension $\dim_{\mathrm{M}}(X, d : \mathbb{C})$. This is a much more local problem than constructing an ε-embedding. Probably we should also emphasize that metric mean dimension is *not* a completely local quantity. The term $\sup_{x\in X}$ in the formula (2.2) has a global nature; we need a uniform estimate all over X. This requires us a detailed quantitative study of X, and it will be the main technical issue of the paper.

Remark 2.6. Some readers might think that the proof of the Lindenstrauss–Weiss theorem (Theorem 2.4) produces an ε-embedding from the information of metric mean dimension. This is true. But it does not provide a *concrete* method to construct an ε-embedding. The proof of [11, Theorem 4.2] is based on a probabilistic argument, and we cannot figure out a deterministic algorithm from it.

2.2. **Energy density.** Here we prepare some facts on the energy density (1.2). The main result (Lemma 2.7 below) is a slightly tricky exchange of the supremum and limit in the definition of $\rho(\mathbb{C}P^N)$. For a positive number λ we define $\mathcal{M}_\lambda(\mathbb{C}P^N)$ as the space of λ-Lipschitz (i.e. $|df| \le \lambda$) holomorphic curves $f : \mathbb{C} \to \mathbb{C}P^N$. There is a natural one-to-one correspondence between $\mathcal{M}(\mathbb{C}P^N)$ and $\mathcal{M}_\lambda(\mathbb{C}P^N)$:

$$\mathcal{M}(\mathbb{C}P^N) \to \mathcal{M}_\lambda(\mathbb{C}P^N), \quad f(z) \mapsto f(\lambda z). \tag{2.3}$$

For $f \in \mathcal{M}_\lambda(\mathbb{C}P^N)$ we define its energy density $\rho(f)$ by

$$\rho(f) := \lim_{n\to\infty} \left(\frac{1}{\mathrm{Area}(\Omega_n)} \sup_{a\in\mathbb{C}} \int_{a+\Omega_n} |df|^2 dxdy \right),$$

where $\{\Omega_n\}_{n\ge 1}$ is a Følner sequence in $\mathbb{C}$. This limit exists because of the Ornstein–Weiss lemma (Lemma 2.3). In particular

$$\rho(f) = \lim_{R\to\infty} \left(\frac{1}{\pi R^2} \sup_{a\in\mathbb{C}} \int_{|z-a|<R} |df|^2 dxdy \right) = \lim_{L\to\infty} \left(\frac{1}{L^2} \sup_{a\in\mathbb{C}} \int_{a+[0,L]^2} |df|^2 dxdy \right).$$

For a Brody curve $f : \mathbb{C} \to \mathbb{C}P^N$ we have $\rho(f(\lambda z)) = \lambda^2 \rho(f)$. Hence from (2.3)

$$\sup_{f\in\mathcal{M}_\lambda(\mathbb{C}P^N)} \rho(f) = \lambda^2 \rho(\mathbb{C}P^N).$$

In [17, Theorem 1.3] we proved the next lemma.

Lemma 2.7.

$$\lambda^2\rho(\mathbb{C}P^N) = \sup_{f\in\mathcal{M}_\lambda(\mathbb{C}P^N)} \rho(f) = \lim_{L\to\infty}\left(\frac{1}{L^2}\sup_{f\in\mathcal{M}_\lambda(\mathbb{C}P^N)}\int_{[0,L]^2}|df|^2dxdy\right).$$

2.3. **Notations.** Here we gather some frequently used notations.

• The number N (which is the complex dimension of $\mathbb{C}P^N$) is fixed throughout the paper. So we treat it as a universal constant. For two quantities x and y we write

$$x \lesssim y$$

if there exists a universal positive constant C satisfying $x \le Cy$. We also write

$$x \lesssim_{a,b,c,\dots,k} y$$

if there exists a positive constant $C(a,b,c,\dots,k)$ which depends only on the parameters $a,b,c,\dots,k$ satisfying $x \le C(a,b,c,\dots,k)y$.

• For a non-zero $u \in \mathbb{C}^{N+1}$ we denote by $[u]$ the point in $\mathbb{C}P^N$ corresponding to u. For two points $u, v \in \mathbb{C}^{N+1}\setminus\{0\}$, the distance between $[u]$ and $[v]$ with respect to the Fubini–Study metric is given by

$$d_{\mathrm{FS}}([u],[v]) = \frac{1}{\sqrt{\pi}}\arccos\frac{|\langle u,v\rangle|}{|u|\,|v|} \in \left[0, \frac{\sqrt{\pi}}{2}\right].$$

Here $\langle u, v\rangle$ is the standard Hermitian inner product, and we choose the branch of arccos satisfying $\arccos 0 = \pi/2$ and $\arccos 1 = 0$. This is a conceptually nice distance. But it is not convenient for concrete calculations. So we introduce another distance by

$$d([u],[v]) := \sin\left(\sqrt{\pi}d_{\mathrm{FS}}([u],[v])\right) = \frac{|u\wedge v|}{|u|\,|v|},$$

where $u \wedge v \in \Lambda^2(\mathbb{C}^{N+1})$ is the exterior product of u and v. The two distances d_{FS} and d are Lipschitz equivalent. For $u = (1, z)$ and $v = (1, w)$ with $z, w \in \mathbb{C}^N$

$$d([1:z],[1:w]) = \frac{\sqrt{|z-w|^2+|z\wedge w|^2}}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}. \tag{2.4}$$

When $N = 1$, the term $z \wedge w$ vanishes and this is the chord distance between two points z and w in the Riemann sphere $\mathbb{C}\cup\{\infty\}$.

• Let $f, g : \mathbb{C} \to \mathbb{C}P^N$ be two holomorphic curves. For a subset $A \subset \mathbb{C}$ we set

$$\mathbf{d}_A(f,g) := \sup_{z\in A} d(f(z), g(z)).$$

3. Proof of the main theorem

In this section we prove Theorem 1.2. Our proof is based on two propositions (Propositions 3.2 and 3.3) whose proofs occupy the rest of the papers. A crucial ingredient of the proof is the following notion.

Definition 3.1. Let R be a positive number, and let Λ be a subset of $\mathbb{C}$. A holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ is said to be R-nondegenerate over Λ if it satisfies

$$\forall a \in \Lambda : \quad \|df\|_{L^\infty(D_R(a))} \geq 1/R.$$

This is a quantitative version of an old idea of Yosida [20]. In [20] a meromorphic function $f \in \mathcal{M}(\mathbb{C}P^1)$ is said to be of first category if it is R-nondegenerate all over the plane for some $R > 0$. This is equivalent to the condition that the closure of the $\mathbb{C}$-orbit of f in $\mathcal{M}(\mathbb{C}P^1)$ does not contain a constant function. This idea of Yosida played a quite important role in [12] for the proof of the lower bound $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(\mathbb{C}P^N)$. Intuitively speaking, Yosida's condition is a kind of "transversality", and Definition 3.1 corresponds to a "quantitative transversality". The system $\mathcal{M}(\mathbb{C}P^N)$ is non-singular around nondegenerate curves in the sense that if $f \in \mathcal{M}(\mathbb{C}P^N)$ is R-nondegenerate over Λ, then a neighborhood of f (whose size depends on R and Λ) can be described by a first-order deformation technique. This is the reason of the importance of the notion.

Unfortunately some Brody curves are degenerate, and they become singularities for our analysis. Our first main task is to resolve these singularities, and the next proposition establishes a quantitative resolution of singularities for our purpose. Recall the notation: $D_R(\Lambda) = \bigcup_{a\in\Lambda} D_R(a)$ for $R \geq 0$ and $\Lambda \subset \mathbb{C}$.

Proposition 3.2. *There exist $\delta_1 > 0$ and $C_1 > 1$ satisfying the following statement. For any $\lambda > 1$ we can choose $R_1 = R_1(\lambda) > 0$ such that for any $f \in \mathcal{M}(\mathbb{C}P^N)$ and any bounded set $\Lambda \subset \mathbb{C}$ we can construct a map*

$$\Phi : \{g \in \mathcal{M}(\mathbb{C}P^N) | \, \mathbf{d}_{D_{R_1}(\Lambda)}(f, g) \leq \delta_1\} \to \mathcal{M}_\lambda(\mathbb{C}P^N)$$

satisfying the following conditions. (Recall that $\mathcal{M}_\lambda(\mathbb{C}P^N)$ is the space of λ-Lipschitz holomorphic curves.)
(1) $\Phi(f)$ is R_1-nondegenerate over Λ.
(2) Let $g_1, g_2 \in \mathcal{M}(\mathbb{C}P^N)$ with $\mathbf{d}_{D_{R_1}(\Lambda)}(f, g_1) \leq \delta_1$ and $\mathbf{d}_{D_{R_1}(\Lambda)}(f, g_2) \leq \delta_1$. For any $z \in \mathbb{C}$

$$C_1^{-1} d(\Phi(g_1)(z), \Phi(g_2)(z)) \leq d(g_1(z), g_2(z)) \leq C_1 \sup_{|w-z|\leq 3} d(\Phi(g_1)(w), \Phi(g_2)(w)). \tag{3.1}$$

The next proposition is a conclusion of our quantitative study of $\mathcal{M}_\lambda(\mathbb{C}P^N)$ around nondegenerate curves.

Proposition 3.3. *For any $R > 0$ and $0 < \varepsilon < 1$ there exist positive numbers $\delta_2 = \delta_2(R)$, $C_2 = C_2(R)$ and $C_3 = C_3(\varepsilon)$ satisfying the following statement. Let $f \in \mathcal{M}_2(\mathbb{C}P^N)$, and let $\Lambda \subset \mathbb{C}$ be a square of side length $L \geq 1$. Suppose f is R-nondegenerate over Λ. Then*

$$\#_{\mathrm{sep}}\left(\{g \in \mathcal{M}_2(\mathbb{C}P^N) | \, \mathbf{d}_{D_5(\Lambda)}(f, g) \leq \delta_2\}, \mathbf{d}_\Lambda, \varepsilon\right) \leq (C_2/\varepsilon)^{2(N+1)\int_\Lambda |df|^2 dxdy + C_3 L}.$$

Assuming Propositions 3.2 and 3.3, we prove the main theorem.

Proof of Theorem 1.2. The lower bound $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(\mathbb{C}P^N)$ was proved in [12]. So the problem is the upper bound. We introduce a distance on $\mathcal{M}(\mathbb{C}P^N)$ by $\mathrm{dist}(f,g) := \sup_{|z|\leq 1} d(f(z), g(z))$. Because of the unique continuation principle, this is a distance compatible with the compact-open topology. We will prove the upper bound on the metric mean dimension: $\dim_{\mathrm{M}}(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist} : \mathbb{C}) \leq 2(N+1)\rho(\mathbb{C}P^N)$. Then we will get $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq 2(N+1)\rho(\mathbb{C}P^N)$ by the Lindenstrauss–Weiss theorem (Theorem 2.4).

Take $1 < \lambda < 2$ and $0 < \varepsilon < 1$. Let $R_1 = R_1(\lambda) > 0$ be the constant introduced in Proposition 3.2. For R_1 and ε, we take $\delta_2 = \delta_2(R_1)$, $C_2 = C_2(R_1)$ and $C_3 = C_3(\varepsilon) > 0$ (the positive constants introduced in Proposition 3.3).

Take a positive number L and $f \in \mathcal{M}(\mathbb{C}P^N)$. We apply Proposition 3.2 to f and $\Lambda := [-4, L+4]^2$. Then we get the map Φ from $\{g \in \mathcal{M}(\mathbb{C}P^N) | \, \mathbf{d}_{D_{R_1}(\Lambda)}(f,g) \leq \delta_1\}$ to $\mathcal{M}_\lambda(\mathbb{C}P^N)$ satisfying the conditions (1) and (2) in Proposition 3.2. We set $\delta = \delta(R_1) = \min(\delta_1, \delta_2/C_1)$ (where $\delta_1 > 0$ and $C_1 > 1$ are the constants introduced in Proposition 3.2) and consider the ball $B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)})$ of radius δ around f in $\mathcal{M}(\mathbb{C}P^N)$ with respect to $\mathrm{dist}_{D_{R_1+10}([0,L]^2)}$. This is contained in the set $\{g \in \mathcal{M}(\mathbb{C}P^N) | \, \mathbf{d}_{D_{R_1}(\Lambda)}(f,g) \leq \delta_1\}$. For $g \in B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)})$, by (3.1)

$$\mathbf{d}_{D_5(\Lambda)}(\Phi(f), \Phi(g)) \leq C_1 \mathbf{d}_{D_5(\Lambda)}(f,g) \leq C_1 \mathrm{dist}_{D_{R_1+10}([0,L]^2)}(f,g) \leq \delta_2.$$

Hence $\Phi(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)}))$ is contained in $\{g \in \mathcal{M}_2(\mathbb{C}P^N) | \, \mathbf{d}_{D_5(\Lambda)}(\Phi(f), g) \leq \delta_2\}$. Then by applying Proposition 3.3 to $\Phi(f)$ and $\Lambda = [-4, L+4]^2$ (note that $\Phi(f)$ is R_1-nondegenerate over Λ),

$$\#_{\mathrm{sep}}(\Phi(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)})), \mathbf{d}_\Lambda, \varepsilon/(3C_1)) \leq (3C_1C_2/\varepsilon)^{2(N+1)\int_\Lambda |d\Phi(f)|^2 dxdy + C_3(L+8)}.$$

By (3.1), for $g_1, g_2 \in B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)})$

$$\mathrm{dist}_{[0,L]^2}(g_1, g_2) \leq C_1 \mathbf{d}_\Lambda(\Phi(g_1), \Phi(g_2)).$$

Hence $\mathrm{dist}_{[0,L]^2}(g_1,g_2) > \varepsilon/3$ implies $\mathbf{d}_\Lambda(\Phi(g_1), \Phi(g_2)) > \varepsilon/3C_1$. By (2.1) and Lemma 2.1

$$\begin{aligned}\#(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)}), \mathrm{dist}_{[0,L]^2}, \varepsilon) &\leq \#_{\mathrm{sep}}(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)}), \mathrm{dist}_{[0,L]^2}, \varepsilon/3) \\ &\leq \#_{\mathrm{sep}}(\Phi(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)})), \mathbf{d}_\Lambda, \varepsilon/(3C_1)) \\ &\leq (3C_1C_2/\varepsilon)^{2(N+1)\int_\Lambda |d\Phi(f)|^2 dxdy + C_3(L+8)}.\end{aligned}$$

Hence the supremum of $\log \#(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)}), \mathrm{dist}_{[0,L]^2}, \varepsilon)$ over $f \in \mathcal{M}(\mathbb{C}P^N)$ is bounded by (note $\Phi(f) \in \mathcal{M}_\lambda(\mathbb{C}P^N)$)

$$(\mathrm{const}_{R_1} + |\log \varepsilon|) \left(2(N+1) \sup_{g \in \mathcal{M}_\lambda(\mathbb{C}P^N)} \int_{[0,L]^2} |dg|^2 dxdy + \mathrm{const}_\varepsilon \cdot L + \mathrm{const}_\varepsilon\right).$$

By Lemma 2.7

$$\lim_{L\to\infty} \left(\frac{1}{L^2} \sup_{g \in \mathcal{M}_\lambda(\mathbb{C}P^N)} \int_{[0,L]^2} |dg|^2 dxdy\right) = \lambda^2 \rho(\mathbb{C}P^N).$$

Therefore

$$\limsup_{L\to\infty} \frac{\sup_{f\in\mathcal{M}(\mathbb{C}P^N)} \log \#(B_\delta(f, \mathrm{dist}_{D_{R_1+10}([0,L]^2)}), \mathrm{dist}_{[0,L]^2}, \varepsilon)}{L^2}$$
$$\leq (\mathrm{const}_{R_1} + |\log \varepsilon|) 2(N+1)\lambda^2 \rho(\mathbb{C}P^N).$$

Divide this by $|\log \varepsilon|$ and let $\varepsilon \to 0$. Then by the formula (2.2) in Lemma 2.5

$$\dim(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist} : \mathbb{C}) \leq 2(N+1)\lambda^2\rho(\mathbb{C}P^N).$$

Here λ is an arbitrary number between 1 and 2. So we can let $\lambda \to 1$ and conclude $\dim(\mathcal{M}(\mathbb{C}P^N), \mathrm{dist} : \mathbb{C}) \leq 2(N+1)\rho(\mathbb{C}P^N)$. □

4. Blowing up degenerate curves

We prove Proposition 3.2 in this section. Here is the idea. If a holomorphic curve f is degenerate around some point $p \in \mathbb{C}$, then we "blow up" it by gluing a sufficiently concentrated rational curve at the point p. Then the resulting curve $\hat{f}$ is nondegenerate around p. We repeatedly apply this procedure to f until all the degenerate regions are eliminated. This technique was first developed in [12] for a single curve f. Here we need to apply it to a family of holomorphic curves. A main new issue is to analyze the changes of the metric structure of the family under blow-ups. We denote by $\mathcal{H}(\mathbb{C}P^N)$ the set of all holomorphic curves $f : \mathbb{C} \to \mathbb{C}P^N$. The following lemma studies the effect of one blow-up in detail.

Lemma 4.1. *We can choose $0 < \delta_3 < 1$, $R_2 > 0$ and $C_4 > 1$ so that for any $p \in \mathbb{C}$, $q \in \mathbb{C}P^N$ and $R \geq R_2$ there exists a map*

$$\Psi : \{f \in \mathcal{H}(\mathbb{C}P^N) | \, f(D_R(p)) \subset B_{\delta_3}(q)\} \to \mathcal{H}(\mathbb{C}P^N), \quad f \mapsto \hat{f},$$

satisfying the following conditions.
(1) For any holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ with $f(D_R(p)) \subset B_{\delta_3}(q)$ (i.e. $d(q, f(z)) \leq \delta_3$ for all $z \in D_R(p)$), the curve $\hat{f}$ satisfies

$$d(f(z), \hat{f}(z)) \leq \frac{C_4}{|z-p|^3}, \tag{4.1}$$

$$\frac{1}{100} < \left\| d\hat{f} \right\|_{L^\infty(D_{R/2}(p))} < 1, \tag{4.2}$$

$$\left| |df|(z) - |d\hat{f}|(z) \right| \leq \frac{C_4}{|z-p|^3}|df|(z) + \frac{C_4}{|z-p|^4} \quad (|z-p| \geq 1). \tag{4.3}$$

(2) Let $f, g : \mathbb{C} \to \mathbb{C}P^N$ be holomorphic curves satisfying $f(D_R(p)) \subset B_{\delta_3}(q)$ and $g(D_R(p)) \subset B_{\delta_3}(q)$. For $|z-p| \leq 1$, the curves $\hat{f}$ and $\hat{g}$ satisfy

$$C_4^{-1} d(\hat{f}(z), \hat{g}(z)) \leq d(f(z), g(z)) \leq C_4 \sup_{|w-p|\leq 2} d(\hat{f}(w), \hat{g}(w)). \tag{4.4}$$

For $|z-p|\geq 1$ *they satisfy*

$$
\begin{aligned}
d(\hat{f}(z),\hat{g}(z)) &\leq \left(1+\frac{C_4}{|z-p|^3}\right) d(f(z),g(z)),\\
d(f(z),g(z)) &\leq \left(1+\frac{C_4}{|z-p|^3}\right) d(\hat{f}(z),\hat{g}(z)).
\end{aligned} \tag{4.5}
$$

Proof. We can assume $p=0$ and $q=[1:0:\cdots:0]$ from the symmetry. The positive constants δ_3, R_2 and C_4 are chosen so that

$$R_2 \gg 1, \quad \delta_3 \ll \frac{1}{R_2^4}, \quad C_4 \gg 1.$$

Specific conditions will be imposed on them through the argument. Take a holomorphic curve $f:\mathbb{C}\to\mathbb{C}P^N$ with $f(D_R)\subset B_{\delta_3}(q)$. We write $f=[1:f_1:\cdots:f_N]$ (f_i: meromorphic function) and set $F(z):=(f_1(z),\dots,f_N(z))$. Then $f(z)=[1:F(z)]$ and the spherical derivative $|df|$ is expressed by

$$|df|(z)=\frac{\sqrt{|F'(z)|^2+|F(z)\wedge F'(z)|^2}}{\sqrt{\pi}(1+|F(z)|^2)}.$$

From $f(D_R)\subset B_{\delta_3}(q)$ with $\delta_3\ll 1$ and $R\geq R_2\gg 1$, we can assume

$$|F(z)|<2\delta_3, \quad |F'(z)|<\delta_3 \quad (|z|\leq R/2). \tag{4.6}$$

Define a holomorphic curve $h:\mathbb{C}\to\mathbb{C}P^N$ by $h(z)=[1:a/z^3:\cdots:a/z^3]$. We choose a positive constant a so that $\|dh\|_{L^\infty(\mathbb{C})}=1/10$. Here we have (set $H(z):=(a/z^3,\dots,a/z^3)$)

$$|dh|(z)=\frac{|H'(z)|}{\sqrt{\pi}(1+|H(z)|^2)}=\frac{3a|z|^2\sqrt{N}}{\sqrt{\pi}(|z|^6+Na^2)}.$$

By $R_2\gg 1$ we can assume $\|dh\|_{L^\infty(D_{R_2/2})}=1/10$. We define a holomorphic curve $\hat{f}:\mathbb{C}\to\mathbb{C}P^N$ by setting $\hat{f}(z):=[1:f_1(z)+a/z^3:\cdots:f_N(z)+a/z^3]$. We set $\hat{F}(z):=F(z)+H(z)$. Then $\hat{f}=[1:\hat{F}]$. We will show that the map $f\mapsto\hat{f}$ satisfies the required conditions.

First, by the distance formula (2.4)

$$d(f(z),\hat{f}(z))=\frac{\sqrt{|H|^2+|F\wedge H|^2}}{\sqrt{1+|F|^2}\sqrt{1+|\hat{F}|^2}}\leq\frac{\sqrt{|H|^2+|F|^2|H|^2}}{\sqrt{1+|F|^2}}=|H|=\frac{a\sqrt{N}}{|z|^3}.$$

This shows (4.1). Next we consider (4.2) and (4.3). We have

$$|d\hat{f}|=\frac{\sqrt{|F'+H'|^2+|F\wedge F'+F\wedge H'+H\wedge F'|^2}}{\sqrt{\pi}(1+|F+H|^2)}.$$

For $|z|\leq R_2/2$, we can assume $|H'(z)|\geq 2\delta_3$ and $|H(z)|\geq 4\delta_3$ by $\delta_3\ll 1/R_2^4$. So by (4.6)

$$\big||F'+H'|-|H'|\big|\leq\delta_3\leq\frac{|H'|}{2}, \quad \big||F+H|-|H|\big|\leq 2\delta_3\leq\frac{|H|}{2} \quad \text{on } D_{R_2/2}. \tag{4.7}$$

Hence $|F'+H'| \geq |H'|/2$ and $|F+H| \leq 2|H|$. Then

$$|d\hat{f}| \geq \frac{|H'|/2}{\sqrt{\pi}(1+4|H|^2)} \geq \frac{|dh|}{8}.$$

This implies a half of (4.2): $\left\|d\hat{f}\right\|_{L^\infty(D_{R_2/2})} \geq \|dh\|_{L^\infty(D_{R_2/2})}/8 = 1/80 > 1/100$. For $|z| \leq R_2/2$ we also have $|F'+H'| \leq 2|H'|$, $|F+H| \geq |H|/2$ and

$$|F\wedge F' + F\wedge H' + H\wedge F'| \leq |F||F'| + |F||H'| + |H||F'| \leq 2\delta_3^2 + 2\delta_3|H'| + \delta_3|H| \leq |H'|$$

by (4.6), $\delta_3 \ll 1$ and $R_2|H'(z)| \gtrsim |H(z)| \geq 2\delta_3$. Therefore (recall $|dh| \leq 1/10$)

$$|d\hat{f}|(z) \leq \frac{\sqrt{4|H'|^2+|H'|^2}}{\sqrt{\pi}(1+|H|^2/4)} \leq \frac{4\sqrt{5}|H'|}{\sqrt{\pi}(1+|H|^2)} = 4\sqrt{5}|dh| < 1.$$

Thus $1/100 < \left\|d\hat{f}\right\|_{L^\infty(D_{R_2/2})} < 1$. (Note that we have not yet finished to prove (4.2).)

From the triangle inequality

$$\left|\sqrt{|F'+H'|^2 + |F\wedge F' + F\wedge H' + H\wedge F'|^2} - \sqrt{|F'|^2+|F\wedge F'|^2}\,\right| \\ \leq \sqrt{|H'|^2 + |F\wedge H' + H\wedge F'|^2} \leq |H'| + |F||H'| + |H||F'|.$$

As we have

$$\frac{1+|F|^2}{1+|\hat{F}|^2} = \frac{1+|\hat{F}|^2 - 2\mathrm{Re}\langle \hat{F}, H\rangle + |H|^2}{1+|\hat{F}|^2} = 1 + \frac{-2\mathrm{Re}\langle \hat{F}, H\rangle + |H|^2}{1+|\hat{F}|^2},$$

we get

$$\left|\frac{1}{1+|\hat{F}|^2} - \frac{1}{1+|F|^2}\right| \leq \frac{|H|+|H|^2}{1+|F|^2}. \tag{4.8}$$

Then by a straightforward calculation, we can check that for $|z| \geq 1$ (note $|H'| \lesssim |H| \lesssim 1$)

$$\Big||d\hat{f}|(z) - |df|(z)\Big| \lesssim |H|\,|df|(z) + |H'|.$$

This shows (4.3). For $R_2/2 \leq |z| \leq R/2$ we have $|df|(z) \lesssim \delta_3$ and

$$|d\hat{f}|(z) \lesssim (1+|H|)\delta_3 + |H'| \ll 1 \quad (\text{by } \delta_3 \ll 1 \text{ and } R_2 \gg 1).$$

Therefore we have $|d\hat{f}| < 1$ over $R_2/2 \leq |z| \leq R/2$. Combining this with $1/100 < \|df\|_{L^\infty(D_{R_2/2})} < 1$, we get (4.2).

Next we consider (2). Let $f = [1 : F]$ and $g = [1 : G]$ be two holomorphic curves with $f(D_R) \subset B_{\delta_3}(q)$ and $g(D_R) \subset B_{\delta_3}(q)$. We have $\hat{f} = [1 : \hat{F}] = [1 : F+H]$, $\hat{g} = [1 : \hat{G}] = [1 : G+H]$ and

$$d(\hat{f}(z), \hat{g}(z)) = \frac{\sqrt{|\hat{F}-\hat{G}|^2 + |\hat{F}\wedge\hat{G}|^2}}{\sqrt{1+|\hat{F}|^2}\sqrt{1+|\hat{G}|^2}},$$

$$\hat{F} - \hat{G} = F - G, \quad \hat{F}\wedge\hat{G} = F\wedge G + (F-G)\wedge H.$$

Suppose $|z| \leq 1$. We have $|\hat{F}| \geq |H|/2$ and $|\hat{G}| \geq |H|/2$ by (4.7). Then

$$d(\hat{f}(z), \hat{g}(z)) \leq \frac{\sqrt{|F-G|^2 + |F \wedge G|^2} + |F-G|\,|H|}{1+|H|^2/4} \leq 2\sqrt{|F-G|^2 + |F \wedge G|^2}$$
$$\lesssim d(f(z), g(z)) \quad (|F(z)|, |G(z)| \leq 2\delta_3 \text{ by (4.6)}).$$

This proves the first part of (4.4). For $|z| \leq 1$ the Cauchy estimate shows

$$|F(z) - G(z)| \lesssim \sup_{|w|=2} |F(w) - G(w)| = \sup_{|w|=2} |\hat{F}(w) - \hat{G}(w)|,$$

$$|F(z) \wedge G(z)| \lesssim \sup_{|w|=2} |F(w) \wedge G(w)| \leq \sup_{|w|=2} \left(|\hat{F}(w) \wedge \hat{G}(w)| + |\hat{F}(w) - \hat{G}(w)|\,|H(w)|\right)$$
$$\lesssim \sup_{|w|=2} \sqrt{|\hat{F}(w) - \hat{G}(w)|^2 + |\hat{F}(w) \wedge \hat{G}(w)|^2} \quad (\text{by } H(w) \lesssim 1 \text{ on } |w| = 2).$$

Since $|\hat{F}|, |\hat{G}| \lesssim 1$ on $|w| = 2$, we get

$$d(f(z), g(z)) \leq |F(z) - G(z)| + |F(z) \wedge G(z)| \lesssim \sup_{|w|=2} d(\hat{f}(w), \hat{g}(w)).$$

This finishes the proof of (4.4).

Suppose $|z| \geq 1$. By (4.8)

$$\frac{1}{1+|\hat{F}|^2} \leq \frac{1+\text{const}\,|H|}{1+|F|^2}, \quad \frac{1}{1+|\hat{G}|^2} \leq \frac{1+\text{const}\,|H|}{1+|G|^2}.$$

As we have

$$\sqrt{|\hat{F}-\hat{G}|^2 + |\hat{F} \wedge \hat{G}|^2} \leq \sqrt{|F-G|^2 + |F \wedge G|^2} + |F-G|\,|H|$$
$$\leq (1+|H|)\sqrt{|F-G|^2 + |F \wedge G|^2},$$

$$d(\hat{f}(z), \hat{g}(z)) \leq (1+\text{const}\,|H|)^2 \frac{\sqrt{|F-G|^2 + |F \wedge G|^2}}{\sqrt{1+|F|^2}\sqrt{1+|G|^2}} \leq (1+\text{const}\,|H|) d(f(z), g(z)).$$

Here we have used $|H| \lesssim 1$. Similarly we get $d(f(z), g(z)) \leq (1+\text{const}\,|H|) d(\hat{f}(z), \hat{g}(z))$. This proves (4.5). We have finished to prove the lemma. □

We restate Proposition 3.2 for the convenience of readers.

Proposition 4.2 (= Proposition 3.2)**.** *For any $\lambda > 1$ we can choose $R_1 = R_1(\lambda) > 0$ such that for any $f \in \mathcal{M}(\mathbb{C}P^N)$ and any bounded set $\Lambda \subset \mathbb{C}$ we can construct a map*

$$\Phi : \{g \in \mathcal{M}(\mathbb{C}P^N) | \, \mathbf{d}_{D_{R_1}(\Lambda)}(f, g) \leq \delta_3/4\} \to \mathcal{M}_\lambda(\mathbb{C}P^N)$$

satisfying the following conditions.
(1) $\Phi(f)$ is R_1-nondegenerate over Λ.
(2) Let $g_1, g_2 \in \mathcal{M}(\mathbb{C}P^N)$ with $\mathbf{d}_{D_{R_1}(\Lambda)}(f, g_1) \leq \delta_3/4$ and $\mathbf{d}_{D_{R_1}(\Lambda)}(f, g_2) \leq \delta_3/4$. For any $z \in \mathbb{C}$

$$d(\Phi(g_1)(z), \Phi(g_2)(z)) \lesssim d(g_1(z), g_2(z)) \lesssim \sup_{|w-z|\leq 3} d(\Phi(g_1)(w), \Phi(g_2)(w)).$$

Proof. We can assume $1 < \lambda < 2$. Let $R = R(\lambda) > R_2$ (R_2 is the constant introduced in Lemma 4.1) be a large positive number. We choose R so large that the conditions (4.9), (4.10) and (4.11) below are satisfied. We will choose R_1 sufficiently larger than R later. Let $\{p_1, \dots, p_n\}$ be a maximal subset of Λ satisfying $|p_i - p_j| > 2R$ ($i \neq j$). Then Λ is contained in the union of the disks $D_{2R}(p_i)$. Since $R \gg 1$, we can assume that for all $z \in \mathbb{C}$

$$\sum_{p_i : |z-p_i|>R} \frac{C_4}{|z-p_i|^3} < \frac{\delta_3}{2}, \tag{4.9}$$

$$\left(1 + \sum_{p_i:|z-p_i|>R/2} \frac{C_4}{|z-p_i|^4}\right) \prod_{p_i:|z-p_i|>R/2} \left(1 + \frac{C_4}{|z-p_i|^3}\right) < \lambda < 2, \tag{4.10}$$

$$\begin{aligned} &\min\left(\frac{\delta_3}{4R\sqrt{\pi}}, \frac{1}{100}\right) \prod_{p_i:|p_i-z|>R} \left(1 - \frac{C_4}{|z-p_j|^3}\right) \\ &\quad - \left(\sum_{p_i:|z-p_i|>R} \frac{C_4}{|z-p_i|^4}\right) \prod_{p_i:|z-p_i|>R} \left(1 + \frac{C_4}{|z-p_i|^3}\right) > \frac{\delta_3}{10R}. \end{aligned} \tag{4.11}$$

Here $0 < \delta_3 < 1$ and $C_4 > 1$ are the constants introduced in Lemma 4.1.

We denote by X the set of $g \in \mathcal{M}(\mathbb{C}P^N)$ satisfying $\mathbf{d}_{D_{R(\Lambda)}}(f, g) \leq \delta_3/4$. We call the point p_i good if $\|df\|_{L^\infty(D_R(p_i))} \geq \delta_3/(4R\sqrt{\pi})$. Otherwise we call it bad. If p_i is bad, then $\|df\|_{L^\infty(D_R(p_i))} < \delta_3/(4R\sqrt{\pi})$ and hence $f(D_R(p_i)) \subset B_{\delta_3/4}(f(p_i))$. This implies

$$\forall g \in X : \quad g(D_R(p_i)) \subset B_{\delta_3/2}(f(p_i)) \quad \text{for bad } p_i. \tag{4.12}$$

We will blow up each curve $g \in X$ around all bad points p_i by using Lemma 4.1 repeatedly.

We inductively construct maps $\Phi_i : X \to \mathcal{H}(\mathbb{C}P^N)$ for $0 \leq i \leq n$ satisfying

$$d(g(z), \Phi_i(g)(z)) \leq \sum_{j=1}^{i} \frac{C_4}{|z-p_j|^3} \quad (\forall g \in X, z \in \mathbb{C}). \tag{4.13}$$

We define Φ_0 as the identity map. Suppose we have already constructed Φ_i. If p_{i+1} is good, then we set $\Phi_{i+1} := \Phi_i$. If p_{i+1} is bad, then we proceed as follows. For $g \in X$ and $z \in D_R(p_{i+1})$

$$d(g(z), \Phi_i(g)(z)) \leq \sum_{j=1}^{i} \frac{C_4}{|z-p_j|^3} < \frac{\delta_3}{2} \quad \text{by (4.9) and (4.13).}$$

Then by (4.12) the image of $D_R(p_{i+1})$ under the map $\Phi_i(g)$ is contained in $B_{\delta_3}(f(p_{i+1}))$. From Lemma 4.1 we can construct a map

$$\Psi : \{h \in \mathcal{H}(\mathbb{C}P^N) | \, h(D_R(p_{i+1})) \subset B_{\delta_3}(f(p_{i+1}))\} \to \mathcal{H}(\mathbb{C}P^N), \quad h \mapsto \hat{h},$$

satisfying the conditions (1) and (2) of Lemma 4.1. Then we set $\Phi_{i+1} := \Psi \circ \Phi_i$. This satisfies the following. Let $g \in X$.

- $d(\Phi_i(g)(z), \Phi_{i+1}(g)(z)) \leq C_4/|z-p_{i+1}|^3$. (Then Φ_{i+1} satisfies the condition (4.13). So we can continue the induction process.)
- $1/100 < \|d\Phi_{i+1}(g)\|_{L^\infty(D_{R/2}(p_{i+1}))} < 1$.
- For $|z-p_{i+1}| \geq 1$

$$\big||d\Phi_{i+1}(g)| - |d\Phi_i(g)|\big| \leq \frac{C_4}{|z-p_{i+1}|^3}|d\Phi_i(g)| + \frac{C_4}{|z-p_{i+1}|^4}. \tag{4.14}$$

- Let $g_1, g_2 \in X$. For $|z-p_{i+1}| \leq 1$

$$C_4^{-1} d(\Phi_{i+1}(g_1), \Phi_{i+1}(g_2)) \leq d(\Phi_i(g_1), \Phi_i(g_2)) \leq C_4 \sup_{|w-p_{i+1}|\leq 2} d(\Phi_{i+1}(g_1)(w), \Phi_{i+1}(g_2)(w)). \tag{4.15}$$

For $|z-p_{i+1}| \geq 1$

$$\begin{aligned} d(\Phi_{i+1}(g_1), \Phi_{i+1}(g_2)) &\leq \left(1 + \frac{C_4}{|z-p_{i+1}|^3}\right) d(\Phi_i(g_1), \Phi_i(g_2)), \\ d(\Phi_i(g_1), \Phi_i(g_2)) &\leq \left(1 + \frac{C_4}{|z-p_{i+1}|^3}\right) d(\Phi_{i+1}(g_1), \Phi_{i+1}(g_2)). \end{aligned} \tag{4.16}$$

We set $\Phi := \Phi_n : X \to \mathcal{H}(\mathbb{C}P^N)$ and $R_1 := 10R/\delta_3$. We will check that they satisfy the required properties. Take $g \in X$. For any $z \in \mathbb{C}$, by (4.10)

$$|d\Phi(g)|(z) \leq \left(1 + \sum_{p_i:|z-p_i|>R/2} \frac{C_4}{|z-p_i|^4}\right) \prod_{p_i:|z-p_i|>R/2} \left(1 + \frac{C_4}{|z-p_i|^3}\right) < \lambda.$$

Here we have used (4.14) and $\|d\Phi_i(g)\|_{L^\infty(D_{R/2}(p_i))} < 1$ for bad p_i. Therefore the image of Φ is contained in $\mathcal{M}_\lambda(\mathbb{C}P^N)$.

For each p_i we can find $z \in D_R(p_i)$ satisfying $|df|(z) \geq \delta_3/(4R\sqrt{\pi})$ (if p_i is good) or $|d\Phi_i(f)|(z) > 1/100$ (if p_i is bad). Then by (4.11)

$$\begin{aligned} |d\Phi(f)|(z) \geq & \min\left(\frac{\delta_3}{4R\sqrt{\pi}}, \frac{1}{100}\right) \prod_{p_j:|p_j-z|>R} \left(1 - \frac{C_4}{|z-p_j|^3}\right) \\ & - \left(\sum_{p_j:|z-p_j|>R} \frac{C_4}{|z-p_j|^4}\right) \prod_{p_j:|z-p_j|>R} \left(1 + \frac{C_4}{|z-p_j|^3}\right) > \frac{\delta_3}{10R}. \end{aligned}$$

This shows that $\Phi(f)$ is R_1-nondegenerate over Λ because $R_1 = 10R/\delta_3 > 10R$ and Λ is contained in the union of $D_{2R}(p_i)$.

Let $g_1, g_2 \in X$. For any $z \in \mathbb{C}$, by (4.15) and (4.16)

$$\begin{aligned} d(\Phi(g_1)(z), \Phi(g_2)(z)) &\leq C_4 \prod_{p_i:|z-p_i|>1} \left(1 + \frac{C_4}{|z-p_i|^3}\right) d(g_1(z), g_2(z)) \\ &\leq 2C_4(1+C_4) d(g_1(z), g_2(z)), \quad \text{(by (4.10))}, \end{aligned}$$

$$\begin{aligned} d(g_1(z), g_2(z)) &\leq \sup_{w:|w-z|\leq 3} \left\{ C_4 \prod_{p_i:|w-p_i|>1} \left(1 + \frac{C_4}{|w-p_i|^3}\right) d(\Phi(g_1)(w), \Phi(g_2)(w)) \right\} \\ &\leq 2C_4(1+C_4) \sup_{w:|w-z|\leq 3} d(\Phi(g_1)(w), \Phi(g_2)(w)) \quad \text{(by (4.10))}. \end{aligned}$$

We have finished the proof. □

5. Quantitative study of nondegenerate curves

In this section we study a neighborhood of a nondegenerate curve by a deformation technique and prove Proposition 3.3. Deformation theory itself is a well-established subject, but here we encounter two unorthodox issues:

- We need thorough quantitative descriptions.
- The complex plane $\mathbb{C}$ is non-compact. So we need to "project" the problem to a compact setting.

Subsection 5.1 is an analytic preparation for the first issue. The second issue is dealt with in Subsection 5.2. We prove Proposition 3.3 in Subsection 5.3.

5.1. Analysis of the operator $(-\Delta + 1)$.

In this subsection we prepare some facts relating to the operator $-\Delta + 1 = -\partial^2/\partial x^2 - \partial^2/\partial y^2 + 1$. The following lemma was proved in Matsuo–Tsukamoto [12, Sublemma 4.4]

Lemma 5.1. *Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a C^2 function such that φ and $\Delta\varphi$ are bounded. Then*

$$\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4 \|(-\Delta+1)\varphi\|_{L^\infty(\mathbb{C})}.$$

Let ψ be a real valued function in $\mathbb{C}$. We define the C^k norm $\|\psi\|_{C^k}$ as $\sum_{i=0}^k \|\nabla^i \psi\|_{L^\infty(\mathbb{C})}$. For $R > 0$ and a subset $\Lambda \subset \mathbb{C}$ the function ψ is said to be R-nondegenerate over Λ if

$$\forall a \in \Lambda : \quad \sup_{D_R(a)} \psi \geq 1/R^2.$$

Lemma 5.2. *For any positive numbers K and R there exists $\kappa = \kappa(K, R) > 0$ satisfying the following statement. Let φ be a real valued bounded C^2 function in $\mathbb{C}$. Suppose $\psi := (-\Delta + 1)\varphi$ is a C^1 function and satisfies*

$$\psi \geq -\kappa, \quad \|\psi\|_{C^1} \leq K, \quad \psi \text{ is } R\text{-nondegenerate all over the plane.}$$

Then $\inf_{z\in\mathbb{C}} \varphi(z) \geq \kappa$.

Proof. Suppose the statement is false. Then for any $n \geq 1$ there exists a bounded C^2 function φ_n in $\mathbb{C}$ such that $\inf \varphi_n < 1/n$ and $\psi_n := (-\Delta + 1)\varphi_n$ satisfies

$$\psi_n \geq -1/n, \quad \|\psi_n\|_{C^1} \leq K, \quad \psi_n \text{ is } R\text{-nondegenerate over } \mathbb{C}.$$

By translation, we can assume $\varphi_n(0) < \inf \varphi_n + 1/n < 2/n$. From Lemma 5.1, the Schauder estimate [7, Theorem 6.2] and $\|\psi_n\|_{C^1} \leq K$, the functions φ_n are bounded in $C^{2,\alpha}$ $(0 < \alpha < 1)$ over every compact subset of $\mathbb{C}$. By the Arzela–Ascoli theorem we can choose a sequence $n_1 < n_2 < n_3 < \dots$ such that φ_{n_k} and ψ_{n_k} converge in C^2 and C^0 respectively over every compact subset of $\mathbb{C}$. We denote their limits by φ and ψ. They satisfy

$$\varphi(0) = \inf \varphi \le 0, \quad (-\Delta + 1)\varphi = \psi, \quad \psi \ge 0, \quad \psi \text{ is } R\text{-nondegenerate over } \mathbb{C}.$$

φ achieves the non-positive minimum at the origin. The condition $(-\Delta+1)\varphi \ge 0$ and the strong minimum principle [7, Theorem 3.5] imply that φ must be a non-positive constant. But then $\psi = (-\Delta + 1)\varphi = \varphi$ cannot be R-nondegenerate over $\mathbb{C}$. □

Let $\Gamma \subset \mathbb{C}$ be a lattice. We study the operator $-\Delta + 1$ over the torus $\mathbb{C}/\Gamma$. It is very important that all the estimates in the lemmas below do not depend on Γ. Indeed we can establish more general statements over the universal cover $\mathbb{C}$. But we don't need them.

Lemma 5.3. *Let K and R be positive numbers. Let ψ be a real valued C^1 function over the torus $\mathbb{C}/\Gamma$ satisfying*

$$\psi \ge -\kappa(K, R), \quad \|\psi\|_{C^1} \le K, \quad \psi \text{ is } R\text{-nondegenerate over } \mathbb{C}/\Gamma.$$

(The last condition means that the pull-back of ψ to the plane is R-nondegenerate over $\mathbb{C}$.) Then there exists a C^2 function φ over the torus satisfying

$$(-\Delta + 1)\varphi = \psi, \quad \|\varphi\|_{C^2} \lesssim_K 1, \quad \varphi \ge \kappa(K, R).$$

Proof. If we find a C^2 function φ in $\mathbb{C}/\Gamma$ satisfying $(-\Delta + 1)\varphi = \psi$, then it satisfies $\|\varphi\|_{C^2} \lesssim_K 1$ (by Lemma 5.1 and the Schauder estimate) and $\varphi \ge \kappa$ (by Lemma 5.2). So the remaining problem is only to solve the equation, and this can be done by a standard L^2 technique. Consider a bounded linear map:

$$L^2_1(\mathbb{C}/\Gamma) \to \mathbb{R}, \quad \phi \mapsto (\psi, \phi)_{L^2}.$$

By the Riesz representation theorem, we can find L^2_1 function φ satisfying $(\nabla\varphi, \nabla\phi)_{L^2} + (\varphi, \phi)_{L^2} = (\psi, \phi)_{L^2}$ for all $\phi \in L^2_1$. Then φ is a distributional solution of $(-\Delta + 1)\varphi = \psi$. By the L^p estimate [7, Theorem 9.11], φ is in L^p_3 for any $1 < p < \infty$. By the Sobolev embedding [7, Corollary 7.11], φ is in C^2. □

So far we have considered the operator $-\Delta + 1$ acting on functions. We will also need the case of a vector bundle coefficient. Let $\mathcal{E}$ be a holomorphic vector bundle over the torus $\mathbb{C}/\Gamma$ with a Hermitian metric. We denote its canonical connection by ∇ and define the curvature $\Theta := [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar{z}}]$. We assume a positivity of Θ; there exists a positive number σ satisfying

$$\forall u \in \mathcal{E} : \langle \Theta u, u\rangle \ge \sigma |u|^2.$$

Let $\bar\partial : \Omega^0(\mathcal{E}) \to \Omega^{0,1}(\mathcal{E})$ be the Dolbeault operator, and $\bar\partial^* : \Omega^{0,1}(\mathcal{E}) \to \Omega^0(\mathcal{E})$ its formal adjoint. The operator $\bar\partial\bar\partial^*$ is an analogue of $-\Delta + 1$ because of the Bochner formula

$$\bar\partial\bar\partial^* a = \frac{1}{2}\nabla^*\nabla a + \Theta a, \quad (a \in \Omega^{0,1}(\mathcal{E})). \tag{5.1}$$

The next lemma is an analogue of Lemma 5.1 and proved in [16, Proposition 4.2].

Lemma 5.4. *Let $a \in \Omega^{0,1}(\mathcal{E})$ be in C^2. Then*

$$\|a\|_{L^\infty(\mathbb{C}/\Gamma)} \leq \frac{8}{\sigma} \left\|\bar\partial \bar\partial^* a\right\|_{L^\infty(\mathbb{C}/\Gamma)}.$$

Lemma 5.5. *Suppose a positive number K satisfies*

$$\|\Theta\|_{C^1_\nabla} := \|\Theta\|_{L^\infty(\mathbb{C}/\Gamma)} + \|\nabla\Theta\|_{L^\infty(\mathbb{C}/\Gamma)} \leq K.$$

Then for any $b \in L^2_2(\Omega^{0,1}(\mathcal{E}))$ there exists $a \in L^2_4(\Omega^{0,1}(\mathcal{E}))$ satisfying

$$\bar\partial\bar\partial^* a = b, \quad \|a\|_{L^\infty(\mathbb{C}/\Gamma)} + \|\nabla a\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim_{\sigma,K} \|b\|_{L^\infty(\mathbb{C}/\Gamma)}.$$

Proof. By the Bochner formula and the positive curvature condition, we can find $a \in L^2_4(\Omega^{0,1}(\mathcal{E}))$ satisfying $\bar\partial\bar\partial^* a = b$. This is the same as the proof of Lemma 5.3. The problem is to show the required estimate. We have $\|a\|_{L^\infty} \lesssim_\sigma \|b\|_{L^\infty}$ by Lemma 5.4. We want to estimate $\|\nabla a\|_{L^\infty}$. By considering a finite covering, we can assume that the injectivity radius of the torus $\mathbb{C}/\Gamma$ is greater than 3. Take a point $p \in \mathbb{C}/\Gamma$. From the condition $\|\Theta\|_{C^1_\nabla} \leq K$, we can find a trivialization g of $\mathcal{E}$ (as a Hermitian C^∞ vector bundle) over the disk $D_2(p)$ such that the connection matrix A representing ∇ under g satisfies

$$\|A\|_{C^1(D_2(p))} \lesssim_K 1.$$

Such g can be constructed by using the parallel translation along the geodesics from p. Under the trivialization g, the operator $\bar\partial\bar\partial^*$ is expressed as

$$\bar\partial\bar\partial^* = \frac{-1}{2}\Delta + B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3,$$

where the L^∞ norms of the coefficients B_1, B_2, B_3 over $D_2(p)$ are $\lesssim_K 1$. Then by using the Sobolev embedding and the L^3 estimate

$$\begin{aligned}
\|\nabla a\|_{L^\infty(D_1(p))} &\lesssim_K \|a\|_{L^3_2(D_1(p))} \quad \text{(Sobolev embedding)}\\
&\lesssim_K \|a\|_{L^3(D_2(p))} + \|b\|_{L^3(D_2(p))} \quad (L^3 \text{ estimate})\\
&\lesssim_\sigma \|b\|_{L^\infty(\mathbb{C}/\Gamma)} \quad \text{(Lemma 5.4)}.
\end{aligned}$$

This holds for every point p. So we get $\|\nabla a\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim_{K,\sigma} \|b\|_{L^\infty(\mathbb{C}/\Gamma)}$. $\square$

5.2. **Constructing holomorphic sections over the torus.** Let E be a holomorphic vector bundle over the plane $\mathbb{C}$ with a Hermitian metric h. Let ∇ and Θ be the canonical connection and curvature of (E, h) respectively. We denote by N the rank of E and regard it as a universal constant.

Lemma 5.6. *Suppose a positive number K satisfies $\|\Theta\|_{C^1_\nabla} \leq K$. For any positive number ε there exists a natural number $m = m(K, \varepsilon)$ satisfying the following statement. Let $\square$ be* *a unit square in the plane. We can find m points $p_1, \dots, p_m$ in $\square$ so that every C^1 section u of E over $D_1(\square)$ satisfies*

$$\|u\|_{L^\infty(\square)} \leq \max_{1\leq i\leq m} |u(p_i)| + \varepsilon \left(\|u\|_{L^\infty(D_1(\square))} + \left\|\bar{\partial} u\right\|_{L^\infty(D_1(\square))} \right).$$

Proof. Take a small positive number $\delta = \delta(K, \varepsilon)$. Let $\{p_1, \dots, p_m\} \subset \square$, $m = m(\delta)$, be a δ-dense subset (i.e. $\square$ is contained in the union of $D_\delta(p_i)$). From the Sobolev embedding $L^3_1(\mathbb{R}^2) \hookrightarrow C^{0,1/3}(\mathbb{R}^2)$ ([7, Theorem 7.17]),

$$\|u\|_{L^\infty(\square)} \leq \max_{1\leq i\leq m} |u(p_i)| + \mathrm{const} \cdot \delta^{1/3} \|\nabla u\|_{L^3(D_{1/2}(\square))}.$$

As in the proof of Lemma 5.5, the L^3 estimate implies

$$\|\nabla u\|_{L^3(D_{1/2}(\square))} \lesssim_K \|u\|_{L^\infty(D_1(\square))} + \left\|\bar{\partial} u\right\|_{L^\infty(D_1(\square))}.$$

We choose $\delta \ll \tau^3$. Then we get the statement. $\square$

Let L be a positive number and set $\Lambda := [0, L]^2$. We consider the following condition.

Condition 5.7. There exist positive numbers K, R and a non-negative C^1 function ϕ in the plane such that

- $\|\Theta\|_{C^1_\nabla} \leq K$ and $\|\phi\|_{C^1} \leq K$.
- $\langle \Theta u, u\rangle \geq \phi(z)|u|^2$ for all $z \in \mathbb{C}$ and $u \in E_z$.
- ϕ is R-nondegenerate over the square Λ.

Let $\Gamma := (L+8)(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ be the lattice in $\mathbb{C}$ generated by $L+8$ and $(L+8)\sqrt{-1}$. We consider the torus $\mathbb{C}/\Gamma$.

Lemma 5.8. *There exists a positive number $L_1 = L_1(K, R)$ such that if $L \geq L_1$ and Condition 5.7 holds then we can construct a Hermitian holomorphic vector bundle $\mathcal{E}$ over the torus $\mathbb{C}/\Gamma$ and a linear map $T : C^\infty((-1, L+1)^2, E) \to H^0(\mathbb{C}/\Gamma, \mathcal{E})$ (a map from the space of C^∞ sections of E over $(-1, L+1)^2$ to the space of holomorphic sections of $\mathcal{E}$) satisfying the following conditions.*
(1)

$$\left|\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{E}) - \frac{1}{\pi}\int_\Lambda \mathrm{tr}(\Theta)dxdy\right| \lesssim_K L.$$

(2) For any $u \in C^\infty((-1, L+1)^2, E)$

$$\|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim_{K,R} \|u\|_{L^\infty((-1,L+1)^2)} + \left\|\bar{\partial} u\right\|_{L^\infty((-1,L+1)^2)},$$

$$\|u\|_{L^\infty(\Lambda)} \lesssim_{K,R} \|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} + \|u\|_{L^\infty((-1,L+1)^2\setminus\Lambda)} + \left\|\bar{\partial} u\right\|_{L^\infty((-1,L+1)^2)}.$$

Proof. Let α be a smooth non-decreasing function in $\mathbb{R}$ such that $\alpha(t) = t$ over $-1 \leq t \leq L+1$, $\alpha'(t) = 0$ over $\{t \leq -2\} \cup \{t \geq L+2\}$ and $|\alpha^{(k)}| \lesssim_k 1$ for $k \geq 1$. We define $\Phi : \mathbb{C} \to \mathbb{C}$ by $\Phi(x, y) := (\alpha(x), \alpha(y))$. Let (E', h') be the pull-back of (E, h) by the map

Φ. This is a C^∞ Hermitian vector bundle over the plane. By Condition 5.7, the curvature of the pull-back connection $\Phi^*\nabla$ satisfies

$$\langle \Theta_{\Phi^*\nabla} u, u\rangle \geq \alpha'(x)\alpha'(y)\phi(\alpha(x),\alpha(y))|u|^2, \quad (u \in E'_{(x,y)}).$$

$\Phi^*\nabla$ is flat outside of $[-2, L+2]^2$. We can classify flat unitary connections over the complement of $[-2, L+2]^2$ by their holonomy maps $\pi_1(\mathbb{C}\setminus[-2,L+2]^2) \to U(N)$. Then there exists a trivialization g of (E', h') outside of $[-2, L+2]^2$ such that the connection matrix A of $\Phi^*\nabla$ with respect to g satisfies $|A| \lesssim 1/L$. (More precisely we can assume that A has the form $A = adx$ with $|a| \lesssim 1/L$.) We take a cut-off $\beta : \mathbb{C} \to [0,1]$ such that $\beta = 1$ on $[-2, L+2]^2$ and $\beta = 0$ outside of $[-3, L+3]^2$. We define a unitary connection ∇' on E' by $\nabla' := g^{-1}(d + \beta A)$.

We define a C^∞ complex vector bundle $\mathcal{E}$ on $\mathbb{C}/\Gamma$ as follows. We naturally regard the torus $\mathbb{C}/\Gamma$ as the quotient of the square $[-4, L+4]^2$ (the opposite edges are identified). We glue the bundle $E'|_{(-4,L+4)^2}$ and $([-4, L+4]^2 \setminus [-3, L+3]^2)\times\mathbb{C}^N$ by the trivialization g. This provides a C^∞ vector bundle on the torus $\mathbb{C}/\Gamma$. We denote it by $\mathcal{E}$. The metric h' and the connection ∇' descend to the torus and become a metric and a unitary connection on $\mathcal{E}$. Let $\bar\partial_{\mathcal{E}} : \Omega^0(\mathcal{E}) \to \Omega^{0,1}(\mathcal{E})$ be the $(0,1)$-part of the covariant derivative $\nabla' : \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})$. This provides a holomorphic structure on $\mathcal{E}$. The curvature of the connection ∇' satisfies

$$\langle \Theta_{\nabla'} u, u\rangle \geq -\frac{c}{L}|u|^2 \quad (u \in \mathcal{E})$$

outside of $[-2, L+2]^2$. Here c is a universal constant. We define a function ψ in the torus $\mathbb{C}/\Gamma$ by

$$\psi(x,y) := \alpha'(x)\alpha'(y)\phi(\alpha(x),\alpha(y)) - \frac{c}{L} \quad (-4 \leq x, y \leq L+4).$$

This satisfies $\langle \Theta_{\nabla'} u, u\rangle \geq \psi|u|^2$ all over the torus, and

$$\psi \geq -\frac{c}{L}, \quad \|\psi\|_{C^1} \lesssim_K 1.$$

Since ϕ is R-nondegenerate over $[0, L]^2$, if L is sufficiently larger than R^2 then ψ is $(R+10)$-nondegenerate over the torus $\mathbb{C}/\Gamma$. By Lemma 5.3 if L is sufficiently large then we can find $\varphi : \mathbb{C}/\Gamma \to \mathbb{R}$ satisfying

$$(-\Delta + 1)\varphi = 4\psi, \quad \|\varphi\|_{C^2} \lesssim_K 1, \quad \varphi \gtrsim_{K,R} 1.$$

We define a metric on $\mathcal{E}$ by $h_{\mathcal{E}} := e^{-\varphi}h'$. Let $\nabla_{\mathcal{E}}$ and $\Theta_{\mathcal{E}}$ be the canonical connection and curvature of the Hermitian holomorphic vector bundle $(\mathcal{E}, h_{\mathcal{E}})$. We have $\Theta_{\mathcal{E}} = \Theta_{\nabla'} + \Delta\varphi/4 = \Theta_{\nabla'} - \psi + \varphi/4$. This enjoys a positivity:

$$\langle \Theta_{\mathcal{E}} u, u\rangle \geq (\varphi/4)|u|^2 \gtrsim_{K,R} |u|^2 \quad (u \in \mathcal{E}).$$

It also satisfies $\|\Theta_{\mathcal{E}}\|_{C^1_{\nabla_{\mathcal{E}}}} \lesssim_K 1$. Therefore $(\mathcal{E}, h_{\mathcal{E}})$ satisfies the assumption of Lemma 5.5.

So far we have constructed the Hermitian holomorphic vector bundle $(\mathcal{E}, h_{\mathcal{E}})$. Next we will construct a linear map $T : C^\infty((-1, L+1)^2, E) \to H^0(\mathbb{C}/\Gamma, \mathcal{E})$. Take a cut-off $\gamma : \mathbb{C} \to [0,1]$ such that $\gamma = 1$ over $\Lambda = [0, L]^2$ and it is supported in $(-1, L+1)^2$. Note that

$(\mathcal{E}, h_{\mathcal{E}})$ is naturally identified with (E, h) over $(-1, L+1)^2$. Let $u \in C^\infty((-1, L+1)^2, E)$ and consider γu. This is supported in $(-1, L+1)^2$, and then it is naturally identified with a section of $\mathcal{E}$ over the torus. We define a holomorphic section $T(u) \in H^0(\mathbb{C}/\Gamma, \mathcal{E})$ by

$$T(u) := \gamma u - \bar{\partial}^*_{\mathcal{E}}(\bar{\partial}_{\mathcal{E}}\bar{\partial}^*_{\mathcal{E}})^{-1}\left(\bar{\partial}_{\mathcal{E}}(\gamma u)\right). \tag{5.2}$$

Here the inverse $(\bar{\partial}_{\mathcal{E}}\bar{\partial}^*_{\mathcal{E}})^{-1}$ is provided by Lemma 5.5.

We check the conditions (1) and (2). First, by the positivity of the curvature $\Theta_{\mathcal{E}}$, the cohomology $H^1(\mathbb{C}/\Gamma, \mathcal{E})$ vanishes. Then by the Riemann–Roch formula

$$\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{E}) = \int_{\mathbb{C}/\Gamma} c_1(\mathcal{E}) = \frac{1}{\pi}\int_{\mathbb{C}/\Gamma} \mathrm{tr}(\Theta_{\mathcal{E}})dxdy.$$

We have $\Theta_{\mathcal{E}} = \Theta$ over $[-1, L+1]^2$ and $|\Theta_{\mathcal{E}}| \lesssim_K 1$. Hence

$$\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{E}) = \frac{1}{\pi}\int_{\Lambda} \mathrm{tr}(\Theta)dxdy + O(L),$$

where the implicit constant depends only on K. This shows the condition (1).

Second, we look at the formula (5.2) and use Lemma 5.5:

$$\|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim_{K,R} \|u\|_{L^\infty((-1,L+1)^2)} + \left\|\bar{\partial}_{\mathcal{E}}(\gamma u)\right\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim \|u\|_{L^\infty((-1,L+1)^2)} + \left\|\bar{\partial} u\right\|_{L^\infty((-1,L+1)^2)}.$$

This is the first half of the condition (2). The other also follows from Lemma 5.5.

$$\begin{aligned}
\|u\|_{L^\infty(\Lambda)} \le \|\gamma u\|_{L^\infty((-1,L+1)^2)} &\le \|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} + \left\|\bar{\partial}^*_{\mathcal{E}}(\bar{\partial}_{\mathcal{E}}\bar{\partial}^*_{\mathcal{E}})^{-1}\left(\bar{\partial}_{\mathcal{E}}(\gamma u)\right)\right\|_{L^\infty(\mathbb{C}/\Gamma)} \\
&\lesssim_{K,R} \|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} + \left\|\bar{\partial}_{\mathcal{E}}(\gamma u)\right\|_{L^\infty(\mathbb{C}/\Gamma)} \quad \text{(Lemma 5.5)} \\
&\lesssim \|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} + \|u\|_{L^\infty((-1,L+1)^2\setminus\Lambda)} + \left\|\bar{\partial} u\right\|_{L^\infty((-1,L+1)^2)}.
\end{aligned}$$

This finishes the proof. □

5.3. **Proof of Proposition 3.3.** Let $T\mathbb{C}P^N$ be the tangent bundle of the projective space, and let

$$\exp : T\mathbb{C}P^N \to \mathbb{C}P^N$$

be the exponential map with respect to the Fubini–Study metric.

Lemma 5.9. *For any $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following statement. Let $p \in \mathbb{C}$ and $f, g_1, g_2 \in \mathcal{M}_2(\mathbb{C}P^N)$, and suppose*

$$\sup_{z \in D_1(p)} d(f(z), g_i(z)) \le \delta \quad (i = 1, 2).$$

Take $u_i \in C^\infty(D_1(p), f^*T\mathbb{C}P^N)$ with $g_i(z) = \exp_{f(z)} u_i(z)$ and $|u_i| \le \delta$. Then

$$|\bar{\partial} u_1(p) - \bar{\partial} u_2(p)| \le \varepsilon \|u_1 - u_2\|_{L^\infty(D_1(p))}.$$

Proof. We can assume $p = 0$ and $f(p) = [1 : 0 : \cdots : 0]$. Let $[1 : w_1 : \cdots : w_N]$ be the standard coordinate around $f(p)$. Set $w := (w_1, \dots, w_N)$. We choose $\delta > 0$ so small that $f(D_{1/10})$, $g_1(D_{1/10})$ and $g_2(D_{1/10})$ are all contained in $\{[1 : w] \,|\, |w| < 1\}$. For $|w| < 2$ and $v = (v_1, \dots, v_N) \in \mathbb{C}^N$ with $|v| \ll 1$, we set

$$[1 : \zeta] := \exp_{[1:w]}\left(\sum_{n=1}^{N} v_n \frac{\partial}{\partial w_n}\right).$$

ζ can be expressed as $\zeta = w + v + P(w, v)$ with a C^∞ function $P(w, v)$ satisfying $P(w, 0) = 0$ and $\nabla_v P(w, 0) = 0$. By the implicit function theorem we can write v as a function of w and ζ:

$$v = \zeta - w + Q(w, \zeta - w),$$

where $Q(w, \xi)$ is a C^∞ function satisfying

$$Q(w, 0) = 0, \quad \nabla_\xi Q(w, 0) = 0. \tag{5.3}$$

Let $f(z) = [1 : F(z)]$, $g_i(z) = [1 : G_i(z)]$ and $u_i(z) = \sum_{n=1}^{N} v_{in} \partial/\partial w_n$ over $D_{1/10}$. Set $v_i := (v_{i1}, \dots, v_{iN})$. We have $|v_i| \le \delta \ll 1$ and

$$v_i(z) = G_i(z) - F(z) + Q(F(z), G_i(z) - F(z)).$$

Differentiating this,

$$\frac{\partial v_i}{\partial \bar z} = \nabla_w Q(F(z), G_i(z) - F(z)) * F'(z) + \nabla_\xi Q(F(z), G_i(z) - F(z)) * (G_i'(z) - F'(z)).$$

We have $|G_i(p) - F(p)| \lesssim \delta$ and $|G_i'(p) - F'(p)| \lesssim \delta$ by the Cauchy estimate. By (5.3)

$$\begin{aligned} \left|\frac{\partial v_1}{\partial \bar z}(p) - \frac{\partial v_2}{\partial \bar z}(p)\right| &\lesssim \delta |G_1(p) - G_2(p)| + \delta |G_1'(p) - G_2'(p)| \\ &\lesssim \delta \sup_{z \in D_{1/10}} |G_1(z) - G_2(z)| \quad \text{(Cauchy estimate)} \\ &\lesssim \delta \left\| u_1 - u_2 \right\|_{L^\infty(D_{1/10})}. \end{aligned}$$

□

We restate Proposition 3.3.

Proposition 5.10 (= Proposition 3.3)**.** *For any $R > 0$ and $0 < \varepsilon < 1$ there exist positive numbers $\delta_2 = \delta_2(R)$, $C_2 = C_2(R)$ and $C_3 = C_3(\varepsilon)$ satisfying the following statement. Let $f \in \mathcal{M}_2(\mathbb{C}P^N)$, and let $\Lambda \subset \mathbb{C}$ be a square of side length $L \ge 1$. Suppose f is R-nondegenerate over Λ. Then*

$$\#_{\mathrm{sep}}\left(\{g \in \mathcal{M}_2(\mathbb{C}P^N) \,|\, \mathbf{d}_{D_5(\Lambda)}(f, g) \le \delta_2\}, \mathbf{d}_\Lambda, \varepsilon\right) \le (C_2/\varepsilon)^{2(N+1)\int_\Lambda |df|^2 dxdy + C_3 L}.$$

Proof. We can assume $\Lambda = [0, L]^2$ without loss of generality. We define a Hermitian holomorphic vector bundle (E, h) over the plane as the pull-back of the tangent bundle $T\mathbb{C}P^N$ and the Fubini–Study metric by the map f. Let ∇ and $\Theta = [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar z}]$ be its canonical connection and curvature. The Fubini–Study metric is a Kähler–Einstein metric, and its holomorphic bisectional curvature is bounded from below by 2π. Then we have

$$\mathrm{tr}(\Theta) = \pi(N+1)|df|^2, \quad \langle \Theta u, u\rangle \geq \pi |df|^2|u|^2 \quad (u \in E).$$

The function $\phi := \pi|df|^2$ is R-nondegenerate over Λ because of the nondegeneracy of f. As $\mathcal{M}_2(\mathbb{C}P^N)$ is compact, there exists a universal positive constant K satisfying

$$\|\Theta\|_{C^1_\nabla} \leq K, \quad \|\phi\|_{C^1} \leq K.$$

Therefore (E,h) satisfies Condition 5.7. Let $L_1 = L_1(K,R)$ be the positive constant introduced in Lemma 5.8. If $1 \leq L \leq L_1$, then the statement of the proposition is obvious because $C_3(\varepsilon)$ can be chosen arbitrarily large. So we can assume $L \geq L_1$ and use Lemma 5.8. It provides us a Hermitian holomorphic vector bundle $\mathcal{E}$ over the torus $\mathbb{C}/\Gamma = \mathbb{C}/(L+8)(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$ and a linear map $T : C^\infty((-1,L+1)^2, E) \to H^0(\mathbb{C}/\Gamma, \mathcal{E})$ satisfying the following conditions. (Note $\mathrm{tr}(\Theta) = \pi(N+1)|df|^2$.)

- $\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{E}) = (N+1)\int_\Lambda |df|^2 dxdy + O(L)$ where the implicit constant in $O(L)$ is universal.
- For any $u \in C^\infty((-1,L+1)^2, E)$

$$\|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} \lesssim_R \|u\|_{L^\infty((-1,L+1)^2)} + \left\|\bar\partial u\right\|_{L^\infty((-1,L+1)^2)}, \tag{5.4}$$

$$\|u\|_{L^\infty(\Lambda)} \leq C\left(\|Tu\|_{L^\infty(\mathbb{C}/\Gamma)} + \|u\|_{L^\infty((-1,L+1)^2\setminus\Lambda)} + \left\|\bar\partial u\right\|_{L^\infty((-1,L+1)^2)}\right), \tag{5.5}$$

where $C = C(R) > 1$ is a constant depending only on R.

By Lemma 5.6 we can choose points $p_1, \dots, p_M \in [-2, L+2]^2 \setminus \Lambda$ with $M \lesssim_\varepsilon L$ such that every C^1 section u of E over $D_4(\Lambda)$ satisfies

$$\|u\|_{L^\infty((-2,L+2)^2\setminus\Lambda)} \leq \max_{1\leq n\leq M} |u(p_n)| + \varepsilon\left(\|u\|_{L^\infty(\partial_4\Lambda)} + \left\|\bar\partial u\right\|_{L^\infty(\partial_4\Lambda)}\right). \tag{5.6}$$

Here recall that $\partial_4\Lambda$ is the set of $z \in \mathbb{C}$ satisfying $D_4(z) \cap \partial\Lambda \neq \emptyset$.

Take a small positive number δ_2, and let X be the set of $g \in \mathcal{M}_2(\mathbb{C}P^N)$ satisfying $\mathbf{d}_{D_5(\Lambda)}(f(z), g(z)) \leq \delta_2$. We define a map S from X to $H^0(\mathbb{C}/\Gamma, \mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}$ as follows. Let $g \in X$ and write it as $g(z) = \exp_{f(z)} u(z)$ with $u \in C^\infty(D_5(\Lambda), E)$ and $|u| \leq \delta_2$. We set

$$S(g) := (T(u), u(p_1), \dots, u(p_M)).$$

(Strictly speaking, $T(u)$ means $T(u|_{(-1,L+1)^2})$.) We define a norm $|||\cdot|||$ on $H^0(\mathbb{C}/\Gamma,\mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}$ by $|||(v, v_1, \dots, v_M)||| := \|v\|_{L^\infty(\mathbb{C}/\Gamma)} + \max_{1\leq n\leq M}|v_n|$. From (5.4) and Lemma 5.9 we have $|||S(g)||| \lesssim_R \delta_2$. So we can assume that the image of S is contained in the unit ball of $H^0(\mathbb{C}/\Gamma, \mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}$.

Take g_1 and g_2 in X, and let $g_i = \exp_f u_i$ with $u_i \in C^\infty(D_5(\Lambda), E)$ and $|u_i| \le \delta_2$. By the estimate (5.5)

$$\|u_1 - u_2\|_{L^\infty(\Lambda)} \le C\Big(\|Tu_1 - Tu_2\|_{L^\infty(\mathbb{C}/\Gamma)} + \|u_1 - u_2\|_{L^\infty((-1,L+1)^2\setminus\Lambda)} + \left\|\bar\partial u_1 - \bar\partial u_2\right\|_{L^\infty((-1,L+1)^2)}\Big).$$

By Lemma 5.9, we can choose $\delta_2 = \delta_2(R)$ so small that

$$\left\|\bar\partial u_1 - \bar\partial u_2\right\|_{L^\infty((-1,L+1)^2)} \le \frac{1}{2C}\|u_1 - u_2\|_{L^\infty((-2,L+2)^2)}.$$

Then

$$\|u_1 - u_2\|_{L^\infty(\Lambda)} \le 2C\|Tu_1 - Tu_2\|_{L^\infty(\mathbb{C}/\Gamma)} + (2C+1)\|u_1 - u_2\|_{L^\infty((-2,L+2)^2\setminus\Lambda)}.$$

Applying (5.6) to $u = u_1 - u_2$, the term $\|u_1 - u_2\|_{L^\infty((-2,L+2)^2\setminus\Lambda)}$ is bounded by

$$\max_{1\le n\le M}|u_1(p_n) - u_2(p_n)| + \varepsilon\left(\|u_1 - u_2\|_{L^\infty(\partial_4\Lambda)} + \left\|\bar\partial u_1 - \bar\partial u_2\right\|_{L^\infty(\partial_4\Lambda)}\right).$$

By $|u_i| \le \delta_2$ in $D_5(\Lambda)$ and Lemma 5.9, $\|u_1 - u_2\|_{L^\infty(\partial_4\Lambda)} + \left\|\bar\partial u_1 - \bar\partial u_2\right\|_{L^\infty(\partial_4\Lambda)} \lesssim \delta_2$. Thus

$$\|u_1 - u_2\|_{L^\infty(\Lambda)} \le (2C+1)\,\|S(g_1) - S(g_2)\| + \mathrm{const}_R \cdot \delta_2\varepsilon.$$

As $d(g_1(z), g_2(z)) \lesssim |u_1(z) - u_2(z)|$, we can choose $\delta_2 \ll 1$ so that

$$\mathbf{d}_\Lambda(g_1, g_2) \le C'\,\|S(g_1) - S(g_2)\| + \frac{\varepsilon}{2}.$$

Here $C' = C'(R) > 1$ depends only on R. Then by Lemma 2.1

$$\#_{\mathrm{sep}}(X, \mathbf{d}_\Lambda, \varepsilon) \le \#_{\mathrm{sep}}\left(S(X), \|\cdot\|, \frac{\varepsilon}{2C'}\right).$$

$S(X)$ is contained in the unit ball of $H^0(\mathbb{C}/\Gamma, \mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}$. So this is bounded by

$$\#_{\mathrm{sep}}\left(B_1\left(H^0(\mathbb{C}/\Gamma, \mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}\right), \|\cdot\|, \frac{\varepsilon}{2C'}\right) \le \left(\frac{6C'}{\varepsilon}\right)^{\dim_{\mathbb{R}}\left(H^0(\mathbb{C}/\Gamma,\mathcal{E})\oplus\bigoplus_{n=1}^M E_{p_n}\right)}.$$

Here we have used Example 2.2. The real dimension of $H^0(\mathbb{C}/\Gamma, \mathcal{E}) \oplus \bigoplus_{n=1}^M E_{p_n}$ is equal to

$$2\dim_{\mathbb{C}} H^0(\mathbb{C}/\Gamma, \mathcal{E}) + 2NM = 2(N+1)\int_\Lambda |df|^2 dxdy + O(L) + 2NM.$$

Recall that $M \lesssim_\varepsilon L$ and the implicit constant in $O(L)$ is universal. So this proves the proposition. $\square$

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